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On the confining potential in 4D $SU(N_c)$ gauge theory with dilaton

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Abstract. Using the formal analogy between the Dick superstring inspired model and the problem of the building of an Eguchi–Hanson metric in 4D $N = 2$ harmonic superspace (HS), we derive a general formula for the quark–quark interaction potential $V(r)$ including the Dick confining potential. The interquark potential $V(r)$ depends on the dilaton–gluon coupling and may be related to the parameterization of confinement by the quark and gluon vacuum condensates. It is also shown how the axion field may be incorporated in agreement with 10D type IIB superstring requirements. Others features are also discussed.

1 Introduction

The dilaton ϕ and the axion χ are scalar fields predicted by superstring theory [1]. Both of them arise in a natural way in the massless spectrum of 10 dimensional (10D) type IIB superstring theory [1,2] and its lower dimensional compactifications. In the language of 4D gauge theory of the field strength $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu}$, ϕ and χ have very special couplings. The dilaton ϕ couples to the gauge fields through a term $\exp(\phi)F^2$ and the axion χ couples to the topological term. The ϕ and χ fields play a central role in superstring dualities [3], F-theory compactifications [4] and in the derivation of the exact results in 4D $N = 2$ supersymmetric gauge theories [5].

Recently it was observed in [6] that a string inspired coupling of a dilaton ϕ to the 4D $SU(N_c)$ gauge fields $A_\mu = T^a A^a_\mu$, with T^a the $(N_c^2 - 1)$ $SU(N_c)$ generators, yields a phenomenologically interesting potential $V(r)$ for the quark–quark interactions. Following [6,7], this potential is obtained as follows: First we start from the following model for the scalar field–gluon coupling

$$L(\phi, A) = -\frac{1}{4G(\phi)}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2}(\partial_\mu\phi)^2 + W(\phi) + J^a_\mu A^\mu_a. \quad (1)$$

Then we choose $G(\phi)$, the coupling of the scalar field ϕ to the $SU(N_c)$ field strength $F_{\mu\nu}$, and the interacting lagrangian $W(\phi)$ as

$$G(\phi) = \text{const.} + \frac{f^2}{\phi^2}, \qquad W(\phi) = \frac{1}{2}m^2\phi^2, \quad (2)$$

where the parameter f is a scale characterizing the strength of the scalar–gluon coupling and m is the mass of the scalar field ϕ. Next we consider the equations of motion of the fields A_μ and ϕ and solve them for static points like the color source of the current density $J^\mu_a = \rho_a\eta^{\mu 0}$. After some straightforward algebra, we find that the Dick quark interaction potential $V_D(r)$ is given by

$$V_D(r) = \frac{1}{r} - f\sqrt{\frac{N_c}{2(N_c - 1)}}\ln[\exp(2mr) - 1]. \quad (3)$$

Equation (3) is very remarkable since for large values of r it leads to a confining potential $V_D(r) \sim 2fm(2(N_c - 1))^{1/2})r$. In this regard, we will show in this paper that for a general gluon–dilaton coupling $G(\phi)$, the quark interaction potential $V(r)$ reads

$$V(r) = \int \mathrm{d}r \frac{G[\phi(r)]}{r^2}. \quad (4)$$

Such a form of the potential is very attractive. On the one hand it extends the usual Coulomb formula $V_c \sim 1/r$ which is recovered from (4) by taking $G = 1$. Moreover for $G \sim r^2$, which by the way corresponds to a coupling $G(\phi) \sim \phi^{-2}$, and $W(\phi) = (m^2/2)\phi^2$, $m \neq 0$, (4) yields a linearly increasing interquark potential $V \sim r$ having the good behavior to describe the $SU(N_c)$ quarks confinement [6–8]. On the other hand (4) may also be used to describe other non-perturbative effects associated with higher dimension quark and gluon vacuum condensates. Following [8], see also [9], one may extract interesting phenomenological information on the dilaton–gluon coupling $G[\phi]$ by comparing (4) to the Bian–Huang–Shen potential $V_{BHS}(r)$, namely

$$V_{BHS}(r) \sim \frac{1}{r} - \sum_{n\geq 0} C_n r^n, \quad (5)$$

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where the C_n's are related to the quark and gluon vacuum condensates. In fact one can do better if one can put the coupling $G(\phi)$ in the form $G[\phi(r)]$. In this case one can predict the type of vacuum condensates of the $SU(N_c)$ gauge theory which contributes to the quark–quark interaction potential [9]. Thus, although the derivation of the formula (4) for the interquark potential from (1) is by itself an important result, there remain however other steps to be taken before one can exploit (4). As mentioned above, a crucial step is to determine what type of couplings $G(\phi)$ can be put in the form $G[\phi(r)]$. In other words, for what couplings $G(\phi)$ can one solve the equation of motion of the scalar field ϕ? This is a technical problem; without solving it one cannot get $V(r)$. Another step which remains is to show how the effective model (1) may be got from a more fundamental theory. If this is possible, one may for instance justify the mass scale f introduced by hand in (2) and (3). One might also get some information on the axion field couplings and more generally on the moduli of 10D superstrings compactified on six dimensional compact manifolds and especially type IIB on Calabi–Yau threefolds [10]. In trying to explore (4), we have observed some remarkable facts among which we quote the three following:

(1) The functional $G[\phi(r)]$, and then the potential $V(r)$ of (4) may be obtained from the following one dimensional lagrangian:

$$L_{\mathrm{D}} = \frac{1}{2}(y')^2 + r^2 W(y/r) + \frac{\alpha}{2r^2} G(y/r), \qquad (6)$$

where $y = r\phi$, $y' = (\mathrm{d}y/\mathrm{d}r)$and $\alpha = g^2/(16\pi^2) \times (N_c - 1)/(2N_c)$ and where g is the gluon coupling constant. In particular L_{D} reads, for $W(\phi)$ and $G(\phi)$ like in (2),

$$2L_{\mathrm{D}} = (y')^2 + m^2 y^2 + \frac{\mu^2}{y^2}, \qquad (7)$$

where $\mu = \alpha f^2$.

(2) Equation (7) has a striking resemblance to the following harmonic superspace lagrangian L_{EH} used in [11] in the derivation of the 4D Eguchi–Hanson metric

$$2L_{\mathrm{EH}} = (D^{++}\omega)^2 + {m^{++}}^2 \omega^2 + \frac{{\mu^{++}}^2}{\omega^2}. \qquad (8)$$

In this equation, ω is an analytic harmonic superspace (HS) superfield taken to be dimensionless, D^{++} is the HS covariant derivative and m^{++} and μ^{++} are coupling constants. More details on HS tools will be described in Sect. 3. Much more precision can be found in [12]. For the moment note only the formal analogy between y, $\mathrm{d}y/\mathrm{d}r$, m and μ of (7) with ω, $(D^{++}\omega)$, μ^{++} and m^{++} respectively. Both of models (7) and (8) involve hermitian fields with a self-interacting potential proportional to the inverse of the square of the scalar field variable.

(3) The Dick potential (3) is viable only for non-zero mass dilaton field exactly as in 4D $N = 2$ supersymmetric theories where the scalar potential is proportional to the mass eigenvalues of the central charges of the 4D $N = 2$ superalgebra [13,5]. Recall by the way that in 4D $N = 2$ supersymmetric QFT, mass terms are generated by central charges. We shall see in Sects. 3 and 4 that this formal analogy between the Dick model (1) and 4D $N = 2$ QFT's is much deeper since it allows us to derive a new model containing (1) and where the symmetries behind the solvability of the Dick equations as well as the couplings of both the dilaton and axion fields are manifest.

The aim of this paper is to generalize the Dick model (1) by exploiting the formal analogy with 4D $N = 2$ supersymmetric theories formulated in HS [11] and using known 4D $N = 2$ exact results. In addition to the derivation of a new model exhibiting a $U(1)$ gauge invariance, we give an interpretation of the mass scale f, introduced by hand in (2), as a Kähler modulus of a blown-up $SU(2)$ singularity of a Calabi–Yau threefold of type II superstring compactifications. The appearance of the local $U(1)$ symmetry in the analysis of (1)–(3) has a quite interesting consequence as it offers a possibility to incorporate in the game the axion field χ couplings. Recall that in a Dick model as formulated in [6], the role of the topological field χ is ignored. We shall show in Sect. 4 how this field can be incorporated by going to a general gauge other than $\phi = \phi^*$.

The presentation of this paper is as follows: In Sect. 2, we formulate the Dick problem as a one dimensional field theory. In Sect. 3 we give general solutions including those of [6,7]. In Sect. 4, we review briefly the building of the Eguchi Hanson hyper-Kähler metric in harmonic superspace. In Sect. 5 we use the formal analogy between the Eguchi–Hanson model and our one dimensional field theoretical formulation of the Dick problem to determine the dilaton couplings, the axion ones and interpret the mass scale f as a kind of Fayet–Iliopoulos coupling. Our conclusion is given in Sect. 6.

2 The Dick model as a one dimensional field theory

Following [6,7], the analysis of the Coulomb problem of the theory (1) is based on considering a point like static color source which in its rest frame is described by a current $J^\mu_a = g\delta(r) C_a \eta^\mu_0$ where C_a is the expectation value of the $SU(N_c)$ generator for a normalized spinor in the color space. These C_a's satisfy the algebraic identity

$$\sum_{a=1}^{N^2-1} C_a^2 = \frac{(N_c - 1)}{2N_c}. \qquad (9)$$

The next step is to use the residual $SO(3)$ space symmetry, which remains after setting $J^\mu_a = \rho_a \eta^\mu_0$, to rewrite the equations of motion

$$[D_\mu, G^{-1}(\phi) F^{\mu\nu}] = J^\nu,$$

$$\partial_\mu \partial^\mu \phi = \frac{\partial W}{\partial \phi} - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \frac{\partial G^{-1}(\phi)}{\partial \phi}, \tag{10}$$

into a simple form. Indeed setting $F^{0i}_a = -gC_a/(4\pi)\partial_i V$, $\alpha = g^2/(16\pi)(N_c - 1)/(2N_c)$ one finds after some easy algebra

$$\frac{\mathrm{d}V}{\mathrm{d}r} = r^{-2} G[\phi], \qquad (a)$$
$$\Delta\phi = \frac{\partial W}{\partial \phi} + \frac{\alpha}{r^4} \frac{\partial G(\phi)}{\partial \phi}. \qquad (b) \tag{11}$$

Note that (11) have four unknown field quantities; the field ϕ, the interacting color potential $V(r)$, the dilaton–gluon coupling $G(\phi)$ and the ϕ potential W. To solve (11) one has to fix two of them. For example choosing $2W = m\phi^2$ and $G(\phi)$ as in (3), one finds

$$\phi = \phi_\mathrm{D}(r) = r^{-1} \left[\frac{\alpha f}{m} \left(1 - \exp(-2mr)\right) \right]^{1/2}, \tag{12}$$

$$V_\mathrm{D}(r) = \frac{1}{r} - f \sqrt{\frac{N_c}{2(N_c - 1)}} \ln[\exp(2mr) - 1]. \tag{13}$$

In general given $G(\phi)$ and $W(\phi)$, the color potential $V(r)$ can be exactly determined on solving one equation, namely (11b). For later use, let us introduce the new dimensionless field $y = r\phi$ and take the spherical coordinate frame (r, θ, φ) to rewrite the lagrangian (1) as

$$L = -\frac{r^2}{2G(\phi)} F^a_{0r} F^{0r}_a - \frac{r^2}{2} \partial_r \phi \partial^r \phi + r^2 W(\phi) + F^a_{0r} \rho_a. \tag{14}$$

In deriving (14), we have used the stationarity of the color source, the $SO(3)$ symmetry and the identity $\Delta(1/r) = \delta(r)$. Putting this equality back into (14) and using the change of variable $y = r\phi$ together with the conventional notation $y' = \partial_r y$, $\partial^r \phi = r^2 \partial_r \phi$ as well as (9), one gets the lagrangian form (6). Consequently the coupling $G(\phi)$ of (1) appears as a part of interacting potential of the one dimensional field theory (6). From this point of view, the finding of the interquark potential $V(r)$ is equivalent to solve the equation of motion

$$y'' \frac{\partial L_\mathrm{D}}{\partial y'} + y' \frac{\partial L_\mathrm{D}}{\partial y} + \partial_r^{\exp} L_\mathrm{D} = 0. \tag{15}$$

3 Solving the Dick model

First of all observe that the lagrangian (6) including the Dick model (7) is a particular one dimensional field theory of lagrangian

$$L = \frac{1}{2}(y')^2 - U(y, r), \tag{16}$$

where $U(y, r)$ is a priori an arbitrary potential. Though simple, this theory is not easy to solve except in some special cases. A class of solvable models is given by potentials of the form

$$U(y) = \lambda^2 y^{2(n+p)} + \gamma^2 y^{2(q-n)} + \delta y^k, \tag{17}$$

where n, p, q and k are numbers and λ^2 , γ^2 and δ are coupling constants scaling as $(\text{length})^{-2}$. The next thing to note is that (17) has no explicit dependence on r and consequently the following identity usually holds:

$$y'^2 = U + c, \tag{18}$$

where c is a constant. Actually (18) is just an integral of motion which may be solved under some assumptions. Indeed by making appropriate choices of the coupling λ as well as the integral constant c, one may linearize y' in (18) as follows:

$$y' = U_1 + U_2. \tag{19}$$

Once the linearization in y' is achieved and the terms U_1 and U_2 are identified, we can show that the solutions of (18) are classified by the product $U_1 U_2$ and the ratio U_1/U_2. In what follows we discuss briefly some interesting examples. For convenience let us rewrite (18) as

$$y'^2 = w_0 + w_1 + C_0, \tag{20}$$

where $w_0 = m^2 y^2$ and w_1 is the interaction term which we take for the moment to be the Dick interaction that is $w_1 = c_1^2 y^{-2}$, where c_1 is a coupling constant. Starting from (20), it is not difficult to see that there are two possibilities to put it in the form (19).

3.1 First possibility: the Dick solution

This corresponds to taking $w_0 = U_1^2$, that is $U_1 = my$ and $U_2 = c_1 y^{-1}$. Putting this back into (19) one gets the Dick solution given by (12) and (13).

3.2 Second possibility: new solutions

In this case the mass term is related to the product $U_1 U_2$ by

$$U_1 U_2 = \pm \frac{1}{2} m^2 y^2. \tag{21}$$

Equation (21) cannot, however, determine U_1 and U_2 independently as in general the following realizations are all of them candidates:

$$U_1 = \lambda y^{n+p}, \qquad U_2 = \gamma y^{q-n}, \tag{22}$$

where the integers p and q are such that $p + q = 2$ and where $\lambda\gamma = \pm m^2$. A remarkable example corresponds to taking $p + q = 1$. In this case we distinguish two solutions according to the sign of the product of $\lambda\gamma$. For $\lambda\gamma = +m^2$, the solution is

$$y(r) = \left[\frac{1}{\lambda} tan \left(\frac{nmr}{\sqrt{2}} + const. \right) \right]^{1/n}. \tag{23}$$

For $\lambda\gamma = -m^2$, we have

$$y(r) = \left[-\frac{1}{\lambda}\tanh\left(\frac{nmr}{\sqrt{2}} + const.\right)\right]^{1/n}. \quad (24)$$

The solutions (23) and (24) have quite interesting features inherited essentially from the features of the tan and tanh functions. We remark that for $n = 0$ the solution is

$$y(r) = const.\exp\left(\frac{\lambda+\gamma}{\sqrt{2}}r\right). \quad (25)$$

In the end of this section, it should be noted that one can go beyond the above-mentioned solutions which are just special cases of general models involving interactions classified by the following constraint equations:

$$U_1.U_2 \sim y^k, \quad (26)$$

where U_1 and U_2 are as in (19) and k is an integer. For $k = 0$, one gets the Dick model and for $k = 2$ one has solutions described in Sect. 3.2. For general values of k, one has to know moreover the ratio U_1/U_2 in order to work out solutions. For the example where

$$U_1 = \lambda y,$$
$$U_2 = \gamma y^{k-1}, \quad k \quad \text{integer}, \quad (27)$$

one can check, after some straightforward algebra, that the solution of y is just a generalization of (12), that is

$$y_k(r) = [r\phi_{\mathrm{D}}]^{2/(2-k)}. \quad (28)$$

For $k = 0$, one discovers the solution (12).

4 The Eguchi–Hanson HS model

To start recall that the Eguchi–Hanson metric is a vacuum solution of self-dual euclidean four dimensional gravity. It is a Ricci flat hyper-Kähler metric having an $SU(2) \times U(1)$ isometry. There are different, but equivalent, ways of writing this metric. A remarkable way of expressing this metric is by using a local coordinate system exhibiting manifestly the $SU(2) \times U(1)$ symmetry. The element of length $\mathrm{d}s^2$ reads

$$\mathrm{d}s^2 = g_{iajb}\mathrm{d}f_1^{ia}\mathrm{d}f_1^{jb} + k_{iajb}\mathrm{d}f_2^{ia}\mathrm{d}f_2^{jb} + h_{iajb}\mathrm{d}f_2^{ia}\mathrm{d}f_1^{jb}, \quad (29)$$

where the metric factors are given by

$$g_{iajb} = \epsilon_{ab}\epsilon_{ij} - \frac{4f_{2ia}f_{2jb}}{f_1^{kc}f_{1kc} + f_2^{kc}f_{2kc}}, \qquad (a)$$
$$k_{iajb} = \epsilon_{ab}\epsilon_{ij} - \frac{4f_{1ia}f_{1jb}}{f_1^{kc}f_{1kc} + f_2^{kc}f_{2kc}}, \qquad (b) \quad (30)$$
$$h_{iajb} = -\frac{4f_{1ia}f_{2jb}}{f_1^{kc}f_{1kc} + f_2^{kc}f_{2kc}}, \qquad (c)$$

together with the $SU(2)$ isovector constraint

$$\epsilon_{ab}(f_1^{ia}f_2^{jb} + f_1^{ja}f_2^{ib}) - \lambda^{ij} = 0. \quad (31)$$

A tricky way to derive this metric is to use the results of 4D $N = 2$ supersymmetric non-linear σ models. In the harmonic superspace approach where 4D $N = 2$ supersymmetry is manifest, the field theoretical model giving the family of Eguchi–Hanson metrics reads in the superfield language

$$S[\omega] = \frac{1}{2k^2} \times \int \mathrm{d}z^{(-4)}\mathrm{d}u\left[(D^{++}\omega)^2 - m^{++2}\omega^2 - \frac{\lambda^{++2}}{\omega^2}\right]. \quad (32)$$

In this equation $\omega = \omega(x_A, \theta^+, \bar{\theta}^+, u)$ is an analytic HS superfield taken to be dimensionless. $D^{++} = \left(u^{+i}(\partial/\partial u^{-i}) -2\theta^+\sigma^m\bar{\theta}^+\partial_m\right)$ is the HS covariant derivative; $\mathrm{d}z^{-4}$ is the analytic superspace measure with $U(1)$ Cartan charge (-4), and the couplings m^{++} and λ^{++} are given by

$$m^{++} = u_i^+u_j^+m^{ij}, \quad \lambda^{++} = u_i^+u_j^+\lambda^{ij}, \quad (33)$$

where u_i^+ and u_i^- are the harmonic variables parameterizing the $SU(2)/U(1) \approx S^2$ sphere. We shall not use these HS tools; we are only interested in the formal analogy with the Dick problem. This is why we shall give only the necessary material in the following. For more details on the HS method and the derivation of the Eguchi–Hanson metric, see [11]. Note also that the Eguchi–Hanson metric with $SU(2) \times U(1)$ isometry corresponds to $m^{++} = 0$. Metrics with $m^{++} \neq 0$ have a $U(1) \times U(1)$ symmetry and fall in the family of multicenter metrics [14,15]. Let us take $m^{++} = 0$ and sketch the main steps in putting (33) in the form (29)–(31). In fact there are two possible paths one may follow: First, there is a direct method which starts from the superfield equation of motion of the hermitian HS superfield ω,

$$D^{++2}\omega = \frac{\lambda^{++2}}{\omega^3}, \quad (34)$$

and uses the θ-expansion of the superfield ω, that is

$$\omega = \phi + \theta^{+2}M^{(-2)} + \bar{\theta}^{+2}\bar{N}^{(-2)} + \theta^+\partial^m\bar{\theta}^+B_m^{(-2)} + \theta^{+2}\bar{\theta}^{+2}P^{(-4)}, \quad (35)$$

where we have ignored fermions. Then one fixes $N = 2$ supersymmetry partially on shell by eliminating the auxiliary fields $P^{(-4)}$ and $B_m^{(-2)}$. The relevant equations are those corresponding to the projection of (34) along the $\theta^+ = 0$ and $\theta^+\sigma^m\bar{\theta}^+$ directions, i.e.

$$\partial^{++2}\phi = \lambda^{++2}/\phi^3,$$
$$\partial^{++}B_m^{-2} = 2\left(\partial_m - \frac{3}{2}\frac{\lambda^{++2}}{\phi^4}B_m^{(-2)}\right)\phi. \quad (36)$$

The next thing to do is to find the explicit dependence of ϕ and $B_m^{(-2)}$ on the harmonic variables $u_i^\pm$ by solving (36). Then put the solution into (33) once the integrations with respect to θ^+ and $\bar{\theta}^+$ are performed. In other words put the solutions $\phi = \phi(u_i^\pm)$, $B_m^{(-2)} = B_m^{(-2)}(u_i^\pm)$ into the following component field action:

$$S[\omega] \sim \frac{1}{k^2} \times \int dx^4 du[\partial^{++} B_m^{(-2)} \partial^m \phi + \partial^m B_m^{(-2)} \partial^{++} \phi]. \quad (37)$$

The last step is to integrate with respect to the harmonic variables. Once this is done, we get the bosonic part of the 4D $N = 2$ supersymmetric non-linear σ model from which one can read the Eguchi–Hanson metric in the ω representation. The second method, which interests us here, is indirect but it has the merit of being based on HS superfield theory exhibiting manifestly the $SU(2) \times U(1)$ symmetry. The main steps of this approach are as follows:
(1) Instead of working with a real superfield ω, we take a complex superfield ω: $\bar{\omega} \neq \omega$.
(2) Modify the action (33) as

$$S[\omega] \sim \frac{1}{2k^2} \int dz^{(-4)} du[|(D^{++} + iV^{++})\omega|^2 + \lambda^{++} V^{++}], \quad (38)$$

where V^{++} is a $U(1)$ gauge superfield. Equation (38) is invariant under the following gauge transformations of parameter λ.

$$\omega' = \exp(-i\lambda)\omega, \quad V'^{++} = V^{++} + D^{++}\lambda. \quad (39)$$

Note that V^{++} has no kinetic term. It is an auxiliary superfield which can be eliminated through its equation of motion namely

$$2V^{++} = \frac{1}{\omega\bar{\omega}}[i(\bar{\omega} D^{++} \omega - \omega D^{++} \bar{\omega}) - \lambda^{++}]. \quad (40)$$

For the special case where ω is real, $\bar{\omega} = \omega$, (40) reduces to

$$V^{++} = -\lambda^{++}/\omega^2, \quad (41)$$

and consequently the action (38) coincides with (33). Note by the way that the term $\lambda^{++} V^{++}$ is a Fayet–Iliopoulos (FI) coupling.
(3) Rewrite (38) in an equivalent form by using the $O(2)$ notation, i.e. express the complex superfield $\omega = \omega_1 + i\omega_2$ as an $O(2)$ doublet (ω_1, ω_2) and introducing two other auxiliary superfields F_1^{++} and F_2^{++},

$$S[\omega_1, \omega_2, F_1^{++}, F_2^{++}] = \frac{1}{2k^2} \int dz^{(-4)} du[(F_1^{++})^2 + 2F_1^{++} D^{++} \omega_1 + (1 \leftrightarrow 2) \quad (42)$$
$$-V^{++}(\omega_1 F_2^{++} - \omega_2 F_1^{++} + \lambda^{++})].$$

Eliminating V^{++}, F_1^{++} and F_2^{++} and choosing the gauge $\omega_2 = 0$ we reproduce the action S_{EH} (33) with $m^{++} = 0$. The second order action (42) is interesting since it has a manifest SU(2) invariance rotating ω_i and F_i^{++}. To make this invariance more explicit we make the following change for both (ω_1, F_1^{++}) and (ω_2, F_2^{++}):

$$\omega = U_a^- q^{+a}, \qquad F^{++} = U_a^+ q^{+a},$$
$$q^{+a} = \epsilon^{ab} q_b^+, \qquad q_a^+ = (q^+, \bar{q}^+), \qquad \epsilon^{12} = 1. \quad (43)$$

Thus for both ω_1 and ω_2, we have $\omega = U_a^- q_I^{+a}$ with $I = 1, 2$ and so on. Putting this back into (42), we get the following action:

$$S = \frac{1}{2k^2} \int dz^{(-4)} du[\bar{q}_1^+ D^{++} q_1^+ + \bar{q}_2^+ D^{++} q_2^+ + V^{++}(\bar{q}_1^+ q_2^+ + \bar{q}_2^+ q_1^+ + \lambda^{++})]. \quad (44)$$

This action has invariance under the following groups:

(i) the $O(2)$ gauge group acting by

$$\delta q_I^+ = \epsilon_{IJ} \lambda q_J^+, \qquad \delta V^{++} = D^{++}\lambda, \qquad \bar{\lambda} = \lambda; \quad (45)$$

(ii) the $U(1)$ subgroup of the rigid $SU(2)$ automorphism group of supersymmetry that leaves λ^{++} invariant;

(iii) The $SU(2)$ Pauli–Cursey symmetry rotating q_I^+ and $\epsilon_{IJ} \bar{q}_J^+$.

Now starting from the last form of the EH action (42) and solving the $\theta^+ = 0$ and $\theta^+ \sigma^m \bar{\theta}^+$ components of the equations of motion,

$$D^{++} q_I - \epsilon_{IJ} V^{++} q_J^+ = 0,$$
$$\epsilon^{IJ} \bar{q}_I^+ q_J^+ + \lambda^{++} = 0 \quad (46)$$

in the Wess–Zumino gauge, one gets, by following the same lines as described for the direct method, the EH metric (29) and (31).

5 The Dick model revisited

In Sect. 2 we learnt that the Dick problem may be formulated as a one dimensional field theory of a lagrangian L_{D} given by (16), namely

$$2L_{\mathrm{D}} = \left(\frac{dy}{dr}\right)^2 - m^2 y^2 - \frac{\mu^2}{y^2}. \quad (47)$$

In Sect. 3 we have shown that hyper-Kähler metrics of the Eguchi–Hanson family can be derived from the following 4D $N = 2$ supersymmetric model:

$$2L_{\mathrm{EH}} = (D^{++}\omega)^2 - {m^{++}}^2 \omega^2 - \frac{{\mu^{++}}^2}{\omega^2}. \quad (48)$$

In Sect. 4 we have seen that this lagrangian is equivalent to the following first order one, once the auxiliary $U(1)$ gauge superfield V^{++} is eliminated through its equation of motion:

$$2L'_{\mathrm{EH}} = |D^{++}\omega|^2 - {m^{++}}^2 \omega\bar{\omega} - V^{++}(\bar{\omega} D^{++} \omega - \omega D^{++} \bar{\omega} - \mu^{++}) + {V^{++}}^2 \omega\bar{\omega}. \quad (49)$$

This form of L_{EH} may also be transformed into two other forms as shown in (42) and (44). The difference between L_{EH} and L'_{EH} is that in (48) ω is hermitian whereas in (50) ω is complex. As we have seen, we can go from (50) to (48) either by constraining the superfield to be real, that is

$$\omega_2 = 0, \tag{50}$$

or equivalently by keeping $\omega_2 \neq 0$ and working in the Wess–Zumino gauge:

$$D^{++}V^{++} = 0. \tag{51}$$

Equation (51) turns on to be helpful in the derivation of the Eguchi–Hanson metric. Now using the formal analogy between (47) and (48), it is not difficult to see that the L_D lagrangian may be also formulated in terms of the auxiliary fields F and $\bar{F}$ as

$$L_2 = F\bar{F} + \bar{F}\partial y + F\partial\bar{y} + V[\xi + \mathrm{i}(y\bar{F} - \bar{y}F)], \tag{52}$$

where V is a one dimensional $U(1)$ gauge field and ξ is a 1D constant vector explicitly breaking invariance under space translations. Note that in this formulation, the two scalars y_1 and y_2 of the complex field $y = (1/2^{1/2})(y_1 + \mathrm{i}y_2)$ represent, respectively, the dilaton ϕ and the axion χ in agreement with the requirement of F-theory and 10D type IIB superstring and 4D N supersymmetric gauge theory. Eliminating the auxiliary fields F and $\bar{F}$ through their equations of motion, namely

$$\begin{aligned} F &= -(\partial + \mathrm{i}V)y = -\nabla y, \qquad (a) \\ \bar{F} &= -(\partial + \mathrm{i}V)\bar{y} = -\nabla\bar{y}, \qquad (a) \end{aligned} \tag{53}$$

one obtains the following first order lagrangian L_1:

$$L_1 = |\nabla y|^2 + m^2 y\bar{y} + \xi V. \tag{54}$$

Moreover eliminating the auxiliary $U(1)$ gauge field V through its equation of motion

$$V = \frac{-1}{2y\bar{y}}[\xi + \mathrm{i}(\bar{y}\partial y + y\partial\bar{y})], \tag{55}$$

one gets the one dimensional field theory of the dilaton–axion system extending (7) which may be recovered from (54) and (55) by going to the gauge fix $y_2 = \chi = 0$. However to exhibit the effect of the axion field χ, one has to keep $\chi \neq 0$ and one imposes a constraint on the gauge field V that we write as follows:

$$C(V, \partial V) = 0. \tag{56}$$

Using this constraint, the first order lagrangian is no longer invariant under the change $y \to \mathrm{e}^{-\mathrm{i}\psi}y$ and $V \to V + \partial\psi$, where ψ is the $U(1)$ gauge parameter, but as a counterpart one can work out a non-trivial solution for the axion field $\chi = \chi(r)$ by solving the conjugate where the field V should be substituted in the equations of motion (55) $\nabla^2 y = m^2 y$ and its complex conjugate by the value $V_0(r)$ verifying the constraint (56) and satisfying the identity $\partial(\bar{y}yV_0) = 0$. In the end of this study we would like to note that the term ξV appearing in (52) plays a similar role as the Fayet–Iliopoulos term $m^{++}V^{++}$ of the 4D $N = 2$ supersymmetric Eguchi–Hanson model (33). Thus the mass scale f introduced by hand in the Dick model may be viewed, under some assumptions, as the scale of breaking of the $U(1)$ symmetry rotating the dilaton and axion fields. Recall by the way that in general supersymmetric gauge theories with a $U(1)$ gauge invariance, the FI term is generally used to break supersymmetry and/or gauge invariance. The FI couplings are Kähler moduli of the Calabi–Yau threefolds on which 10D type II superstrings are compactified, and their magnitudes are of order of the Calabi–Yau compactification scale.

6 Conclusion

Inspired by the dilaton–gluon coupling in superstring theory, Dick built a field theoretical model having the remarkable property of leading to a confining quark–quark interaction potential. The model is mainly a 4D $SU(N_c)$ gauge theory coupled to a massive scalar field ϕ of lagrangian (1) and a dilaton–gluon coupling $G(\phi) = 1 + f^2/\phi^2$, where f is a mass scale introduced by hand. The parameter f may be compared with the mass scale of the σ model of mesonic theory [16]. The confining phase of Dick model is parameterized by the non-zero mass for the dilaton and the non-vanishing f. In trying to analyze the potential $V_D(r)$ we have observed that the Dick problem has a perfect formal analogy with the problem of building the Eguchi–Hanson metric in 4D $N = 2$ supersymmetric harmonic superspace. This formal similarity appears at several levels. In the Introduction we have quoted some of these striking analogies. For example, vanishing masses for both Dick and Eguchi–Hanson scalar fields lead to trivial potentials. Another example is that the mass scale f which is introduced by hand and interpreted as a compactification scale by Dick plays a similar role as the 4D $N = 2$ FI coupling appearing in the Eguchi–Hanson model, (33). Recall by the way that now it is well established that the FI couplings are of the order of the compactification scale since they are just the Kähler moduli of the Calabi–Yau threefold on which the 10D type II superstrings are compactified. To understand the striking similarity between the Dick problem and the Eguchi–Hanson one, we have reformulated the Dick problem as a one dimensional field theory. As a consequence we have found a general formula for the interquark potential $V(r)$ which of course depends on the nature of the dilaton–gluon $G[\phi]$ as shown in (4). The beauty of this formula is not only that it extends the Coulomb and Dick theory but also that it can be compared with known parameterizations of the confinement, especially the contribution of the quark and gluon vacuum condensates. From this point of view, the Dick model as we have formulated it may be viewed as a phenomenological theory modeling the non-perturbative contributions responsible for confinement. In this regard a more explicit analysis will be presented in [17].

Moreover, having at hand the 1D field theoretical formulation of the Dick model and the analogy with the 4D $N = 2$ Eguchi–Hanson model, we have shown how the axion field may be incorporated in the game in agreement with the requirement of F-theory and 10D type IIB superstrings according to which the dilaton and the axion form a complex field. In our formulation the dilaton–axion model is represented by a 1D $U(1)$ gauge theory of the lagrangian (52). The $U(1) \approx SO(2)$ symmetry rotates the dilaton and the axion fields and allows one to interpret the Dick mass scale as a kind of FI coupling.

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